

What I Learned Teaching AP Calculus

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55th Georgia Math Conference, October 17, 2014



Outline

How I Started

What I Learned

Questions



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How I Started Teaching AP Calculus

I taught students the way I was taught. . .

- ▶ Algebra/Trig review
- ▶ Limits
- ▶ Derivatives
- ▶ Applications of Derivatives
- ▶ Integrals
- ▶ Applications of Integrals
- ▶ BC-only topics



Well What's Wrong With That?

The traditional sequence wasn't working for my students.

- ▶ Was it me?
- ▶ Was it them?
- ▶ Was it the course sequence?
- ▶ Was it their preparation?
- ▶ Was it my preparation?
- ▶ Was it the book?
- ▶ Was it the activities?
- ▶ Was it all of the above?



How Do I Fix It?

I had to really understand

- ▶ why calculus is taught the way it is,
- ▶ the knowledge my students were bringing to the class,
- ▶ why AP includes the topics it does,
- ▶ why AP does not include certain topics, and
- ▶ calculus itself.



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Here are some of the things I learned.



Keep the Algebra Minimized

Compute the following.

1. $\lim_{x \rightarrow 1} \frac{3x^3 + 7x^2 - 2x - 8}{(x - 1)^5}$

2. $\lim_{x \rightarrow 3} \frac{\sqrt{x + 3} - \sqrt{6}}{x - 3}$

3. $\frac{d}{dx} \left(\frac{(2x - 3)^5 (3x - 2)^4}{(x - 4)^3} \right)$

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3. $\frac{d}{dx} \left(\frac{(2x - 3)^5 (3x - 2)^4}{(x - 4)^3} \right)$

Are those more instructive than the following?

1. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$

2. $\lim_{x \rightarrow 1} \frac{(x - 1)^2}{x - 1}$

3. $\frac{d}{dx} \left(\frac{(x - 1)(x - 2)}{x} \right)$



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Limit problems on the AP Exam fall into four categories:



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1. direct evaluation;
2. left- and right-hand limits for piecewise functions;
3. definitions of continuity and the derivative;
4. l'Hôpital's rule.



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Limits mean behavior, and we report behavior with y -values.



Limits As Behavior

The function

$$f(x) = \frac{x^2 - 9}{x - 3}$$

behaves as if it will be equal to 6 when $x = 3$. Of course, x cannot be 3, but its behavior as x approaches 3 is that $f(x)$ should be 6. Hence,

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = 6.$$



Limits As Behavior

This “behavior” explains why

- ▶ limits may not exist;
- ▶ horizontal asymptotes are limits as $x \rightarrow \pm\infty$;
- ▶ we can talk of the behavior of the average rate of change as the instantaneous rate of change (the derivative)



Stop Spending the First Month on Review

- ▶ Summer packets of review material can work, but
 - ▶ if the kids already know the stuff, then it's just busy work;
 - ▶ if the kids don't know the stuff, it just frustrates them and increases their hatred of math; and
 - ▶ the kids still don't see *why* they need to know the stuff!



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 - ▶ the kids still don't see *why* they need to know the stuff!
- ▶ Review as you go when you need to
- ▶ Make review part of each homework assignment (and each test)
- ▶ Reviewing in August sends the signal that this class is like every other math class they've taken



Use Differential Equations To Be Different

The first six days of my class

Day 1 Rates

Day 2 Exponential population growth

Day 3 Logistic population growth

Day 4 Position-velocity problem

Day 5 Euler's method

Day 6 Slope fields



Use Differential Equations To Be Different

Advantages of this approach

- ▶ Wildly different than any math class they've ever taken
- ▶ Opportunity to teach them calculator usage
- ▶ Reinforces that the method is as important as the answer
- ▶ Students who struggled in previous math classes tend to excel
- ▶ Introduces the need for more exact methods, and leads into differentiation
- ▶ Meaning of the derivative is crystal clear
- ▶ Applications drive the concepts



The Point of the AP Curriculum is Approximations

This idea is more obvious with BC than with AB:

- ▶ slope fields
- ▶ Euler's method
- ▶ graph and table problems
- ▶ Taylor polynomials
- ▶ Riemann sums
- ▶ numerous definite integrals only computable on a calculator
- ▶ emphasis on setting up integrals
- ▶ numerous functions which “model” a situation



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A cubic approximator for a function:

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2 + \frac{1}{6}f'''(a)(x - a)^3$$



Only Simplify When It Helps

I consider these expressions simplified:

$$\frac{1}{1 + (x - 3)^2} \quad \frac{7}{\sqrt{12}} \quad \frac{x^3 + 4x^2}{x + 1}$$

The operations one could perform does not help simplify the expression. However, writing

$$\frac{10 + x^2}{5} \quad \text{as} \quad 2 + \frac{1}{5}x^2$$

helps a great deal!



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This is also a correct statement:

$$\lim_{x \rightarrow a} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow a} (x + 3), \quad \text{for all } a$$



Only Simplify When It Helps

Simplification does not necessarily lead to a correct answer!

The domain of

$$f(x) = \sqrt{x^4 - 16x^2}$$

is $(-\infty, -4] \cup \{0\} \cup [4, \infty)$, but simplifying leads to the function

$$f(x) = |x|\sqrt{x^2 - 16},$$

whose domain excludes zero.



Get to Derivatives of Trig Functions ASAP

Differentiating the functions

$$x^2 \sin x \quad \frac{\cos x}{x^2} \quad \sin(x^2)$$

justifies developing product, quotient, and chain rules.
Better justification (and less algebra) than

$$x^2(x+3)^5 \quad \frac{x^2+1}{x^4} \quad (x^2-x-1)^{10}$$



Get to Derivatives of Trig Functions ASAP

Order of presentation:

- ▶ Power Rule; Polynomials
- ▶ Sine and Cosine
- ▶ Product Rule
- ▶ New rule: $\frac{d}{dx} \left(\frac{1}{f(x)} \right) = \frac{-f'(x)}{[f(x)]^2}$; use this to find Secant and Cosecant
- ▶ Quotient Rule; use this to find Tangent and Cotangent
- ▶ Chain Rule



Use “exp” for Exponentials

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How can we get our students to stop making this silly mistake? By never using this notation until later in the course. Instead use the notation $\exp(x)$ for the exponential function.



Use “exp” for Exponentials

Students recognize that differentiating $\cos(x^2)$, $\ln(x^2)$ and $\sin(x^2)$ requires the chain rule.

To students, e^{x^2} looks like a power. With a change in notation, that mistake is corrected:

$$\frac{d}{dx}(\exp(x^2)) = 2x \exp(x^2).$$



Use “exp” for Exponentials

Exponentials with other bases:

$$\exp(x \ln 2) = e^{x \ln 2} = \left(e^{\ln 2}\right)^x = 2^x,$$

so other bases are just transformations of $\exp(x)$.



Hold Off Logarithms and Exponentials ALAP

Wait until after the FTC before dealing with logarithms. Then the function $\ln(x)$ may be developed so students understand it! From that, develop the exponential function.



The Natural Logarithm

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- ▶ is positive for $x > 1$ and negative for $0 < x < 1$;



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- ▶ is positive for $x > 1$ and negative for $0 < x < 1$;
- ▶ is unbounded so its range is all real numbers.



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Consider $L(kx)$ for constant k . Then

$$\frac{d}{dx}[L(kx)] = \frac{d}{dx} \int_1^{kx} \frac{1}{t} dt = \frac{1}{kx} \cdot k = \frac{1}{x}$$

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so that $L(kx)$ is also an antiderivative of $\frac{1}{x}$. Hence, since two antiderivatives can at most differ by a constant, we know that $L(kx) = L(x) + C$. However, when $x = 1$, this becomes $L(k) = L(1) + C$. But we know $L(1) = 0$, so we have $L(k) = C$. Therefore,

$$L(kx) = L(x) + L(k).$$



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Finally, the function L defined by

$$L(x) = \int_1^x \frac{1}{t} dt$$

is called the *logarithm* of x . Note that there must be some value of x such that $L(x) = 1$ and we call this value e .

The Natural Logarithm

What about other bases?

- ▶ We want $f(x) = cL(x) = 1$ for a particular value of x . Call this particular x -value b . Then $cL(b) = 1$, or $c = 1/L(b)$.
- ▶ The number b is called the *base* of the logarithm. Hence,

$$f(x) = c \log_b(x) = \frac{L(x)}{L(b)}.$$

- ▶ Since b and c are related by $cL(b) = 1$, then when $b = e$, we have $c = 1$. Hence, $L(x) = \log_e(x) = \ln(x)$.



The Exponential Function

- ▶ Let $\exp(x)$ be the inverse of $\ln(x)$. Then, by definition, if $\ln(a) = b$, then $a = \exp(b)$.
- ▶ As an inverse, $\exp(x)$ satisfies

$$\exp(\ln(x)) = x \quad \text{and} \quad \ln(\exp(x)) = x.$$

- ▶ Since $\ln(1) = 0$ and $\ln(e) = 1$, then when $x = 0$ and when $x = e$ we get

$$\exp(0) = 1 \quad \text{and} \quad \exp(1) = e.$$



The Exponential Function

- ▶ For reals a , b , and p and positive reals m and n , we let

$$m = \exp(a), \quad n = \exp(b), \quad \text{and} \quad p = \ln(mn).$$

- ▶ Then

$$\ln(m) = a, \quad \ln(n) = b, \quad \text{and} \quad \exp(p) = mn.$$

- ▶ Hence, $p = \ln(mn) = \ln(m) + \ln(n) = a + b$, or $p = a + b$.
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- ▶ Hence, $p = \ln(mn) = \ln(m) + \ln(n) = a + b$, or $p = a + b$.
- ▶ So then $\exp(p) = \exp(a + b)$.
- ▶ Also, $\exp(p) = mn = \exp(a) \exp(b)$.
- ▶ We have two expressions for $\exp(p)$. Equate them:

$$\exp(a + b) = \exp(a) \exp(b).$$



The Exponential Function

What is the derivative of $\exp(x)$? Begin by composing $\ln(x)$ with $\exp(x)$ in two ways.

- ▶ First, $\ln(\exp(x)) = x$.
- ▶ Second, we also have that $\ln(\exp(x)) = \int_1^{\exp(x)} \frac{1}{t} dt$.
- ▶ Therefore, $\int_1^{\exp(x)} \frac{1}{t} dt = x$.
- ▶ Taking derivatives of both sides, we get

$$\frac{1}{\exp(x)} \cdot \exp'(x) = 1, \quad \text{or} \quad \exp'(x) = \exp(x).$$



The Exponential Function

Repeatedly using the property $\exp(a + a) = \exp(a) \exp(a)$ gives us the property $\exp(na) = (\exp(a))^n$.

- ▶ Let $a = 1$ in $\exp(a)^n = \exp(na)$.
- ▶ Recall $\exp(1) = e$.
- ▶ Then $\exp(n) = \exp(1)^n = e^n$.



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- ▶ Recall $\exp(1) = e$.
- ▶ Then $\exp(n) = \exp(1)^n = e^n$.
- ▶ This gives us another way to denote the exponential function $f(x) = \exp(x)$: $f(x) = e^x$.



Use Hyperbolic Functions

Advantages:

- ▶ Reinforces trig derivatives and antiderivatives
- ▶ Interesting applications
- ▶ Expected to know them for Calc II and Calc III
- ▶ In exponential form, they have appeared in the AP Exam



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How did I come to these conclusions?
I stopped listening to College Board.



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Travesty! Recount such heresy before I call the AP Police!



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I realized I needed to study teaching *Calculus* so I could teach *AP Calculus*. So I searched other sources.



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- ▶ Many calculus books: single- and multi-variable, real analysis
- ▶ MAA resources
- ▶ The history of calculus
- ▶ The development of trigonometry and logarithms



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Then create a course *that makes sense* using the AP Course Description as a guide for minimal content



Erase the Word “Unit” from Our Vocabulary

- ▶ Planning by “units” is a convenience of administrators
- ▶ Speaking of “units” reinforces the student belief that math is disconnected, discrete chunks of unrelated material
- ▶ Do college mathematics professors think in “units”?



Write Better Tests

- ▶ Every test is a chance to do AP Exam preparation
 - ▶ Use multiple-choice and free-response
 - ▶ Part with a calculator and part without
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 - ▶ Use old AP free-response on every test
- ▶ Do not use test banks!
- ▶ Do not let anyone else write your calculus tests!
- ▶ Do not allow kids to use their crutches...



Write Better Tests

Write the equation of the line tangent to the graph of $y = x^2 - 5x + 7$ at the point where $x = 3$.

- A) $y = x + 2$
- B) $y = -x + 2$
- C) $y = 2x + 1$
- D) $y = x - 2$
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This is a BETTER multiple-choice question:

The equation of the line tangent to the graph of $y = x^2 - 5x + 7$ at the point where $x = 3$ can be written in the form $y = mx + b$. Compute $m + b$.

- A) -1
- B) 0
- C) 1
- D) 2
- E) 3



Consult Old Books

Find all points on the curve $y^4 - 5y^2 = x^4 - 4x^2$ at which the tangent lines are (a) vertical; (b) horizontal.

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Compute

$$\frac{dy}{dx} = \frac{2x^3 - 4x}{2y^3 - 5y}.$$

Then by the standard procedure there are vertical tangents at $(\pm 2, 0)$ and $(0, 0)$; and horizontal tangents at $(0, \pm\sqrt{5/2})$ and $(0, 0)$.

Consult Old Books

A *singular point* is a point on a curve at which the derivative becomes $0/0$. This indicates the presence of a *double point*—a point through which the curve passes twice. The tangent lines of the curve at a double point are found by considering only those terms of degree two in the equation and solving.

(Clyde Love and Earl Rainville, *Differential and Integral Calculus*, 5th edition, MacMillan, New York, 1954.)



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Find all points on the curve $y^4 - 5y^2 = x^4 - 4x^2$ at which the tangent lines are (a) vertical; (b) horizontal.

Since the terms of degree two are $-5y^2$ and $4x^2$, we ignore the other terms and solve $-5y^2 = -4x^2$ to get $y = \pm 2x/\sqrt{5}$. So the origin has no vertical or horizontal tangents.

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7. Why do we teach students to rationalize denominators?
8. Why do we still teach slope-intercept form?

