Taylor Series, Maclaurin Series, and the Lagrange Form of the Remainder

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Outline

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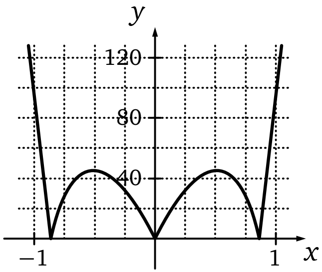
2011 BC #6

Let $f(x) = \sin(x^2) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown.



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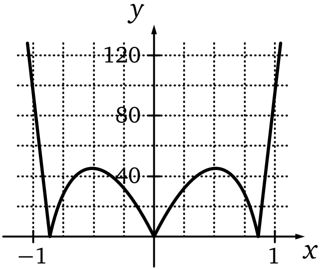

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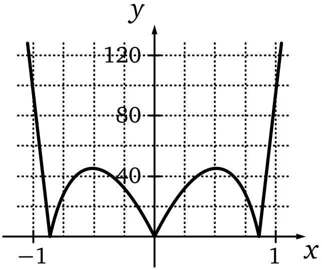

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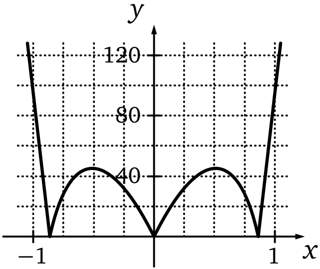

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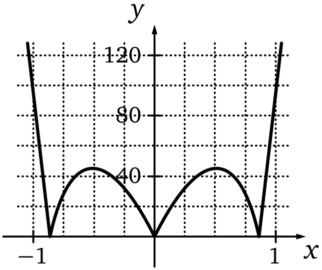
(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.



Taylor Series?

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots$$

Outline

Approximating Polynomials

- Set up Taylor Polynomials (and Taylor Series) earlier in the year
- Extend the tangent line idea to tangent curves (a polynomial used to approximate another function)
- So the basic approximating polynomial is the tangent line

Approximating Polynomials

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Solution.

The tangent line to $f(x) = \sqrt{x}$ centered at $x = 81$ is

$$L(x) = f(81) + f'(81)(x - 81) = 9 + \frac{1}{2\sqrt{81}}(x - 81).$$

Then $f(77) \approx L(77) = 9 + \frac{1}{18}(77 - 81) = 9 - \frac{2}{9} = 8\frac{7}{9}$. □

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$$\begin{aligned} Q'(x) &= f'(a) + 2C(x-a) \\ Q''(x) &= 2C. \end{aligned}$$

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$$Q(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

Approximating Polynomials

By the same logic, a cubic approximator should match the third derivative of f. So we assume the approximation has the form

$$\begin{aligned} B(x) &= L(a) + Q(a) + C(x-a)^3 \\ &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + C(x-a)^3 \end{aligned}$$

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$$B(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3.$$

Sample Problems for Approximating Polynomials

Problem 2

Write the quadratic approximating polynomial for $f(x) = e^{-x}$ *centered at* $x = 0$ *and use it approximate* $e^{-0.2}$.

Sample Problems for Approximating Polynomials

Problem 3

Consider the following table of data for the function f.

x	5.0	5.2	5.4	5.6	5.8
$f(x)$	9.2	8.8	8.3	7.7	7.0

1. *Estimate* $f'(5.2)$.
2. *Using the linear approximator to the graph of* $f(x)$ *at* $x = 5.2$*, approximate the value of* $f(5.26)$.
3. *Is* $f''(5.4)$ *positive or negative? Use this to the determine the concavity of the graph of* f*. Show the work that leads to your answers.*

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- Generate more terms of power series through the notion that $f^{(n)}(a) = n!C$
- In this way, the *idea* of Taylor series is simply a continuation of what came before
- Now put it all together: Taylor series constitute
  - approximating polynomials
  - questions of convergence
  - power series and intervals of convergence
  - differentiation
  - manipulation of series

Taylor Series

Theorem 1 (Taylor's Theorem)

Let f be a function such that $f^{(k+1)}(x)$ exists for all x in the interval $(a-r, a+r)$. Then

$$P_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is the kth degree Taylor polynomial of f at a, and

$$R_k(x) \leq \frac{|f^{(k+1)}(c)|}{(k+1)!}|x-a|^{k+1}$$

is the Lagrange form of the remainder, where c is a number between a and x which maximizes $f^{(k+1)}$. Moreover, assume f has derivatives of all orders. Then $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ if and only if $R_k \to 0$ as $k \to \infty$.

Taylor Series

Problem 4

Use the third-order Taylor polynomial for $f(x) = \ln x$ centered at $x = 1$ to approximate $\ln 1.06$.

Solution.

The Taylor polynomial is

$$P_3(x) = x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

so the approximation is

$$\begin{aligned} P_3(1.06) &= 0.06 - \frac{0.06^2}{2} + \frac{0.06^3}{3} \\ &= 0.06 - 0.0018 + 0.000072 = 0.058272. \quad \square \end{aligned}$$

Taylor Series

This is the Lagrange form of the remainder:

$$R_k(x) \leq \frac{|f^{(k+1)}(c)|}{(k+1)!}|x-a|^{k+1}.$$

Note the remainder is the difference between the function $f(x)$ and the Taylor polynomial $P_k(x)$.

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- This remainder is like the remainder for Alternating Series, in that it is the absolute value of the first unused term;
- Major difference: we maximize the numerator on the interval $[a,x]$ to get the largest bound.

Taylor Series

Problem 5

Determine the error in using the third-order Taylor polynomial for $f(x) = \ln x$ *centered at* $x = 1$ *to approximate* $\ln 1.06$.

Solution.

The remainder is

$$R_3(x) \leq \frac{|f^{(4)}(c)|}{4!}|x-1|^4.$$

The fourth derviative of $\ln x$ is $-6/x^4$ whose maximum value on the interval $[1, 1.06]$ is when $x = 1$; this leads to a numerator of -6. So the error must be less than

$$R_3(1.06) \leq \frac{|-6|}{4!}|0.06|^4 = \frac{0.00001296}{4} = 0.00000324. \quad \square$$

Outline

Manipulation of Series

From the geometric series $\sum x^n = \frac{1}{1-x}$, we may get – on the interval of convergence – power series for

- $\frac{1}{1-x^2}$ by replacing x^2 for x in $\frac{1}{1-x}$
- $\frac{1}{1+x}$ by replacing $-x$ for x in $\frac{1}{1-x}$
- $\frac{1}{1+x^2}$ by replacing $-x^2$ for x in $\frac{1}{1-x}$
- $\frac{1}{(1-x)^2}$ by differentiating $\frac{1}{1-x}$
- $-\ln|1-x|$ by integrating $\frac{1}{1-x}$
- $\arctan x$ by integrating $\frac{1}{1+x^2}$
- $\ln|1+x|$ by integrating $\frac{1}{1+x}$
- $\frac{x^2}{(1+x)^3}$ by differentiating $\frac{1}{1+x}$ twice and then multiplying by x^2.

Manipulation of Series

From the Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

we may find the Maclaurin series for e^{-x^2}:

$$\begin{aligned} e^{-x^2} &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ &= 1 - x^2 + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \end{aligned}$$

Manipulation of Series

Know Maclaurin series for

- $\frac{1}{1-x}$
- $\sin x$
- $\cos x$
- e^x,

and know how to manipulate these to obtain others.

Outline

Free-Response Problems

- 2004 BC #2 (Form B)
- 2008 BC #3
- 2011 BC #6