Solving Some (Possibly AP) Calculus Problems
GCTM Conference 2013

Presenters: Marshall Ransom & Chuck Garner

Objectives:

EXPECT TO:

1. Do more calculus than you will listen to

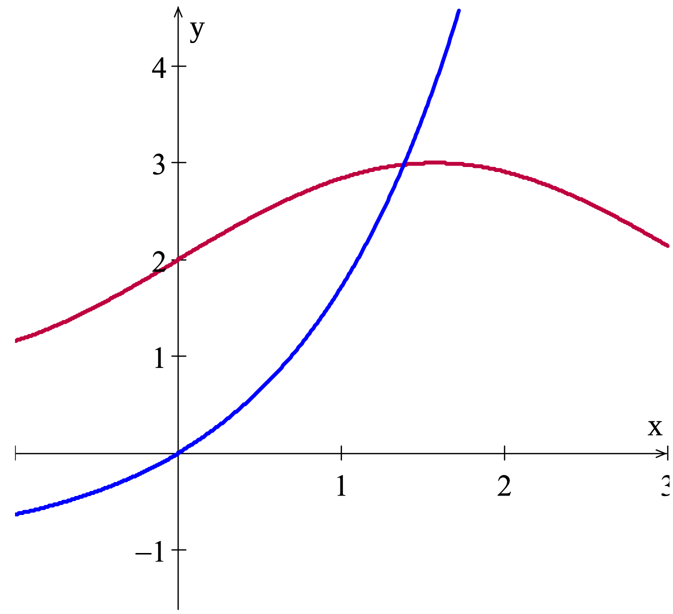
2. Reinforce an appreciation for the difficulties students encounter when learning calculus and solving problems

3. Discuss a variety of calculus topics in the context of solving problems

4. When possible, put solutions into the context of how the problems might be graded on the AP Calculus Exam

5. Provide ideas for the writing of problems and problem-solving in your teaching

(1) A region is bounded by the y-axis, $y = e^x - 1$, and $y = \sin(x) + 2$, as shown in the graph at right. (Units are cm).



(a) Calculate the area of this region.

(b) A solid is formed in the region bounded by the two curves $y = e^x - 1$ and $y = \sin(x) + 2$ and the line $y = 2$. This solid is formed by cross-sections that are squares perpendicular to the y-axis. Calculate the volume of this solid.

(c) A line $y = k$ where $k > 2$ passes horizontally through the region. Set up an equation involving k which if solved would calculate the specific value of k that would divide the volume found in part b in half.

(NOTE: Increasingly, we have seen on the AP test a differential equation in a real world context).

(2) The cooling system in my old truck holds about 10 liters of coolant. Last summer I flushed the system by running tap water into a tap-in on the heater hose while the engine was running and was simultaneously draining the thoroughly mixed fluid from the bottom of the radiator. Water flowed in at about the same rate as the mixture flowed out. This was about 2 liters per minute. The mixture was initially about 50% antifreeze. If we let W be the amount of water in the coolant after t minutes then it follows that

$$\frac{dW}{dt} = 2 - 2\left(\frac{W}{10}\right)$$

(a) Explain why the expression for $\frac{dW}{dt}$ is correct.

(b) Find W as a function of time.

(c) How long should I have let the water run into the system to ensure that the mixture was 95% water?

(3) Let $F(x)$ and $F'(x)$ be continuous and assume values in the table below. Use $F(x)$, $F'(x)$ and the values in the table to answer the questions below.

x	–1	0	1	2	3	4	5
$F(x)$	18	0	–12	0	54	168	360
$F'(x)$	–15	–11	0	30	81	150	237

(a) What is the slope of a line tangent to the graph of $F(x)$ at $x = 0$?

(b) Write an equation of a line tangent to $F(x)$ at $x = 4$.

(c) Why can we not determine whether $F(x)$ is increasing on the interval (2, 3)?

(d) Calculate the value of $\frac{d}{dx}\left(x^3 \bullet F(x)\right)$ at $x = 1$.

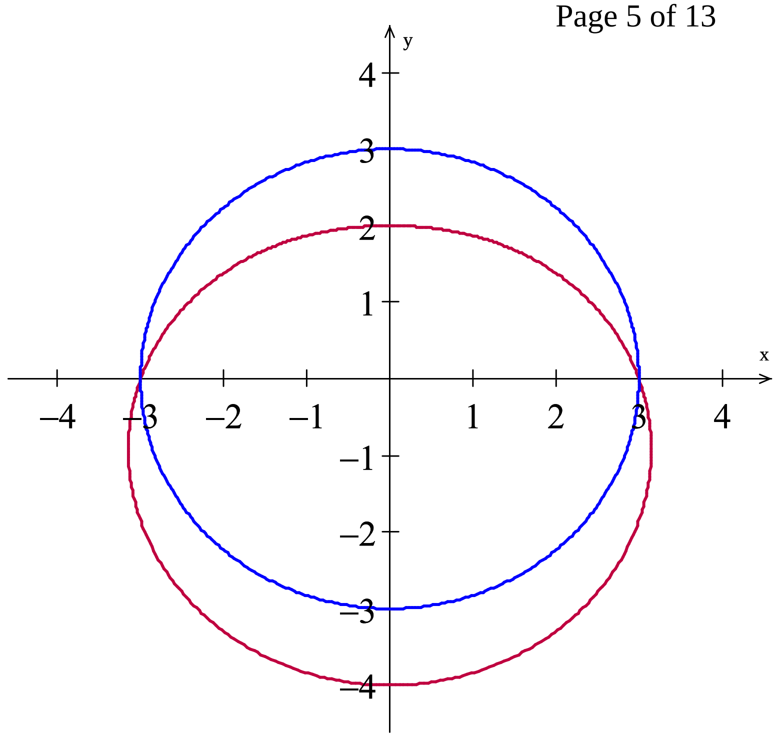
(e) Write and solve your own problem requiring the quotient rule.

(f) Calculate the value of $\int_3^5 F'(x)\,dx$.

(g) Write and solve your own integral problem requiring use of $\tan^{-1}(x)$.

(h) If $H(x) = \int_{x^2}^{2} F(t)\,dt$, calculate $H'(1)$.

(4) The BC question (NOT calculator active): Graphs of the polar curves $r = 3 - \sin(\theta)$ and $r = 3$ are shown in the figure at right. Answer the following questions.



(a) A particle moves along the curve $r = 3 - \sin(\theta)$ such that at any time t in seconds, $\theta = t$. What is the position vector of this particle in terms of t? What are the exact rectangular coordinates of the position when $t = \frac{\pi}{6}$?

(b) What is the exact value of the slope of a line tangent to the curve $r = 3 - \sin(\theta)$ where $\theta = \frac{\pi}{6}$?

(c) Write an equation of the line tangent to $r = 3 - \sin(\theta)$ at the point where $\theta = \frac{\pi}{6}$. Use this equation to estimate the y-coordinate of the point where $x = x_0 - 0.01$, x_0 being your x-coordinate from part a.

(d) What is the area of the region above the x-axis, below the circle and above the limaçon?

(e) The length of the curve $r = 3 - \sin(\theta)$ above the x-axis from points A(3, 0) to B(–3, 0) is approximately 7.7253. What is the approximate average speed of the particle as it traverses this arc starting at $t = 0$?

(4) The AB question (this is NOT calculator active):
Two particles are in motion on the x-axis. The position of particle P is given by $p(t) = 3\sin\left(\frac{\pi}{2}t\right)$ and the position of particle Q is given by $q(t) = t^3 - 2t^2 + t - 3$ for any time $t \geq 0$.

(a) How far apart are the particles at time $t = 0$? Write, but do not evaluate, an expression for the average distance between the particles on the interval $0 \leq t \leq 4$.

(b) On the interval $0 \leq t \leq 3$ find all times t when particle Q is moving to the left.

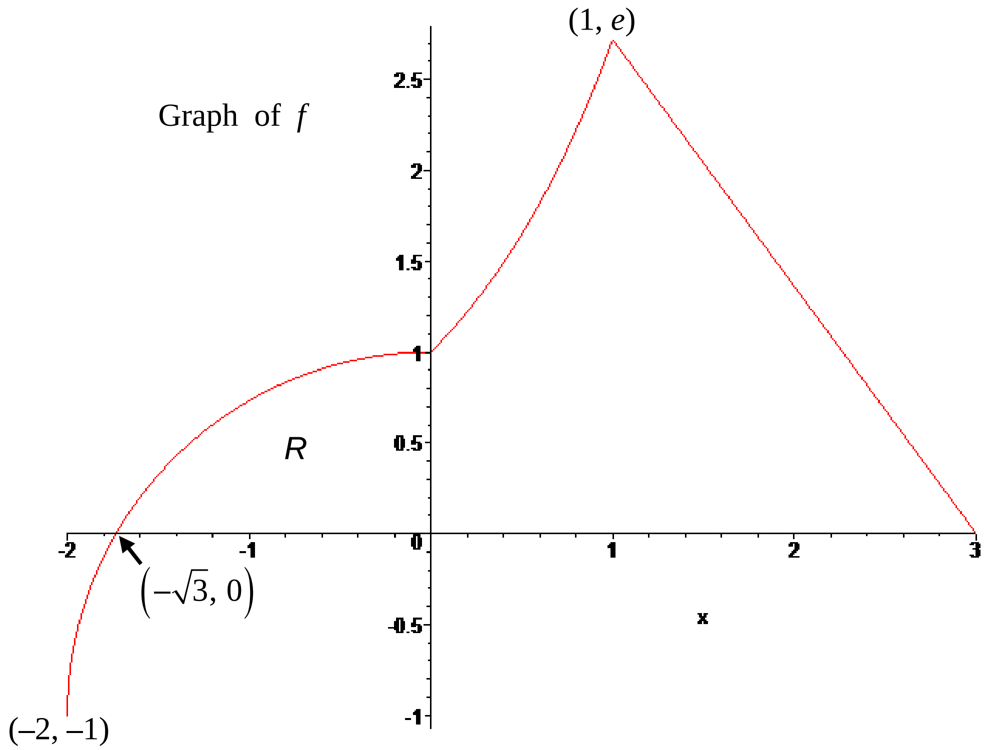
(c) On the interval $0 \leq t \leq 3$ find all times when the particles are moving in the same direction.

(d) What is the velocity of particle Q at time $t = 2$? Is the particle Q speeding up, slowing down, or neither of these at the time $t = 2$? Justify your answer.

(e) Find the total distance traveled by particle P on the interval $0 \leq t \leq 2$. Reminder: this is not a calculator active problem.

(5) The continuous function f is defined for $-2 \le x \le 3$. The graph of f consists of a quarter circle, e^x, and one line segment as shown at right. Answer the following questions using the fact

that $g(x) = \int_0^x f(t)\,dt$.



(a) Calculate the values of $g(1)$ and $g(3)$.

(b) The region R is the portion of the quarter circle above the x-axis. The exact area of this region is given by $-\frac{\sqrt{3}}{2}+\frac{2\pi}{3}$. Using this value, calculate the exact value of $g\left(-\sqrt{3}\right)$.

(c) Explain why $g\left(-\sqrt{3}\right)$ is an absolute minimum value for $g(x)$.

(d) Find the average rate of change of f on the interval $0 \le x \le 3$. There is no value $c \in (0,3)$ for which $f'(c)$ equals that average value. Explain why this does not contradict the Mean Value Theorem.

(e) What is the value of $g''(2)$?

(f) For what value(s) of x does the graph of g have a point of inflection? Justify your answer(s).

Solutions and Comments

(1) Note: This problem would be a calculator active problem, both to find the intersection point and to calculate the numerical value of the integral. For example, the calculation of an anti-derivative in part b would involve integration by parts (a BC topic) and some rather creative persistence. The point of intersection of these curves will sometimes be awarded one point on the AP Calculus Exam if used in work which proceeds to answering the questions. This point is $(x, y) = (1.38183\ldots,\ 2.98219\ldots)$. Remember, 3 digit accuracy is all that is required on the AP Calculus exam. In the solutions below, We will refer to these numbers as A = 1.38183… and B = 2.98219…, using A and B in the integrals.

(a) Area = $\displaystyle\int_0^A \left(\sin(x)+2-\left(e^x-1\right)\right)dx \approx 1.975$.

(b) (On the AP test, students sometimes must show some work in producing x in terms of y.) The values of x are: $x=\ln(y+1)$ and $x=\sin^{-1}(y-2)$. The first value of x shown is the curve on the right.

Volume = $\displaystyle\int_2^B \left(Right-Left\right)^2 dy = \int_2^B \left(\ln(y+1)-\sin^{-1}(y-2)\right)^2 dy \approx 0.5475...$

(c) For a k which finds half the volume, use $\displaystyle\int_2^k \left(\ln(y+1)-\sin^{-1}(y-2)\right)^2 dy \approx \tfrac{1}{2}(0.5475)$

OR use $\displaystyle\int_k^B \left(\ln(y+1)-\sin^{-1}(y-2)\right)^2 dy \approx \tfrac{1}{2}(0.5475)$.

Variations on parts b & c: (b) Use semi-circles and (i) calculate the volume directly and (ii) calculate the volume using an appropriate fraction of your answer to the square cross-section problem.
(c) Assume for $k > 2$, the line $y = k$ is moving upward at a rate of 1.5 cm/min. When $k = 2.5$ at what rate is the volume using square cross-sections increasing (region bounded below by $y = 2$ and above by $y = k$)? Express your answer using proper units.

Variation (b): Using semi-circular cross-sections:

(i) $r=\tfrac{1}{2}\left(\ln(y+1)-\sin^{-1}(y-2)\right) \to r^2=\left(\tfrac{1}{2}\left(\ln(y+1)-\sin^{-1}(y-2)\right)\right)^2$

Therefore the volume $\displaystyle =\int_2^B \tfrac{1}{2}\pi r^2\,dy = \tfrac{\pi}{2}\int_2^B \left(\tfrac{1}{2}\left(\ln(y+1)-\sin^{-1}(y-2)\right)\right)^2 dy \approx 0.215$

(ii) OR fractional method: The ratio of the area of a semi-circle with radius r to the area of a square of side $2r$ is $\tfrac{\pi}{8}$. The volume is $\tfrac{\pi}{8}\times 0.5475 \approx 0.215$.

Variation (c): We know that $\displaystyle V=\int_2^k \left(\ln(y+1)-\sin^{-1}(y-2)\right)^2 dy$. Therefore,

$$\frac{dV}{dt}=\left(\ln(k+1)-\sin^{-1}(k-2)\right)^2\times\frac{dk}{dt} \;\to\; \frac{dV}{dt}=\left(\ln(2.5+1)-\sin^{-1}(2.5-2)\right)^2\times 1.5 \quad \text{cm}^3/\text{min}$$

(2) (a) The rate of change in the amount of water is water inflow rate minus water outflow rate. The water inflow rate is 2 liters per minute. The fraction of water in the outflow is $\frac{W}{10}$ giving an outflow rate of $2\times\frac{W}{10}$.
Therefore inflow rate minus outflow rate = $\frac{dW}{dt} = 2-2\left(\frac{W}{10}\right)$.

(b) We separate and integrate:

$$\frac{dW}{dt} = 2-2\left(\frac{W}{10}\right) = \frac{10-W}{5}$$

$$\int \frac{dW}{10-W} = \int \frac{1}{5}\,dt$$

$$-\ln\left|10-W\right| = \frac{1}{5}t + C \;\leftarrow\; \text{on the AP test, this line alone usually gets at least 3 points}$$

$$10-W = Ke^{-\frac{1}{5}t} \;\leftarrow\; \left(\text{What about absolute value?}\right)$$

$$W = 10-Ke^{-\frac{1}{5}t}$$

The initial condition is $W(0)$ = 50% water = 5 liters.
We get from the last line above $5 = 10-Ke^{-\frac{1}{5}\bullet 0} \;\rightarrow\; K=5$. Therefore $W(t) = 10-5e^{-\frac{1}{5}t}$.

(c) $W(t) = 9.5 = 10-5e^{-\frac{1}{5}t} \;\rightarrow\; 0.5 = 5e^{-\frac{1}{5}t} \;\rightarrow\; \ln(0.1) = -\frac{1}{5}t \;\rightarrow\; t = -5\ln(0.1)$ minutes ≈ 11.513 minutes

(3) (a) $F'(0) = -11$

(b) $F(4) = 168$ and $F'(4) = 150 \;\rightarrow\; y - 168 = 150(x-4)$ (AP test: often, 1 point each for slope and equation)

(c) For $F(x)$ to be increasing on an interval, $F'(x)$ must be positive on the entire interval. In fact, we do not know from the table any values of $F'(x)$ on the open interval (2, 3).

(d) $\frac{d}{dx}\left(x^3 \bullet F(x)\right) = x^3 F'(x) + 3x^2 F(x)$. Therefore, at $x = 1$ we have $1^3 \bullet F'(1) + 3\left(1^2\right)F(1) = 0 + 3(-12)$.

(e) Quotient Rule possible example: $\frac{d}{dx}\left(\frac{x^3}{F(x)}\right) = \frac{F(x)\bullet 3x^2 - x^3 F'(x)}{\left(F(x)\right)^2}$

Choose any value of x from the table and use appropriate $F(x)$ and $F'(x)$ values (beware of $F(x) = 0$).

(f) $\int_3^5 F'(x)\,dx = F(5) - F(3) = 360 - 54$

(g) Possible $\tan^{-1}(x)$ example: $\int_1^3 \frac{F'(x)\,dx}{1 + \left(F(x)\right)^2} = \tan^{-1}\left(F(3)\right) - \tan^{-1}\left(F(1)\right) = \tan^{-1}(54) - \tan^{-1}(-12)$

(h) $H(x) = \int_{x^2}^{2} F(t)\,dt \;\rightarrow\; H'(x) = -F\left(x^2\right)\bullet 2x \;\rightarrow\; H'(1) = -F\left(1^2\right)\bullet 2(1) = -(-12)\bullet 2$

(4) BC

(a) Because $x = r\cos(\theta)$ and $y = r\sin(\theta)$, the position vector is $\left\langle 3\cos(t) - \cos(t)\sin(t),\ 3\sin(t) - \sin^2(t) \right\rangle$.

At $t = \frac{\pi}{6}$ we have $x = 3 \times \dfrac{\sqrt{3}}{2} - \dfrac{1}{2} \times \dfrac{\sqrt{3}}{2} =$ DO NOT SIMPLIFY (it's $\dfrac{5\sqrt{3}}{4}$) and $y = 3 \times \dfrac{1}{2} - \left(\dfrac{1}{2}\right)^2$ OR $\dfrac{5}{4}$

(b) slope $= \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{3\cos(\theta) - 2\sin(\theta)\cos(\theta)}{-3\sin(\theta) - \left(\sin(\theta)(-\sin(\theta) + \cos(\theta)\cos(\theta)\right)}$

At $\theta = \frac{\pi}{6}$ we have a mess or (DO NOT SIMPLIFY) $-\dfrac{\sqrt{3}}{2}$

(c) $y - y_0 = m(x - x_0) \;\rightarrow\; y - \dfrac{5}{4} = -\dfrac{\sqrt{3}}{2}\left(x - \dfrac{5\sqrt{3}}{4}\right)$

At $x = \dfrac{5\sqrt{3}}{4} - 0.01$ we have $y - \dfrac{5}{4} = -\dfrac{\sqrt{3}}{2}\left(\left(\dfrac{5\sqrt{3}}{4} - 0.01\right) - \dfrac{5\sqrt{3}}{4}\right) \;\rightarrow\; y \approx \dfrac{\sqrt{3}}{2}(0.01) + \dfrac{5}{4}$

(d) Area = $\dfrac{1}{2}\displaystyle\int_0^{\pi} r_1^2\, d\theta - \dfrac{1}{2}\int_0^{\pi} r_2^2\, d\theta$ if $r_1 = 3$ and $r_2 = 3 - \sin(\theta)$.

$$\frac{1}{2}\int_0^{\pi}\left(3^2 - \left(3 - \sin(\theta)\right)^2\right)d\theta = \frac{1}{2}\int_0^{\pi}\left(6\sin(\theta) - \sin^2(\theta)\right)d\theta = \frac{1}{2}\left(-6\cos(\theta) - \frac{1}{2}\left(\theta - \frac{\sin(2\theta)}{2}\right)\right)\Bigg|_0^{\pi} = 6 - \frac{\pi}{4}$$

(e) The particle starts its motion at time $t = 0$ and reaches point B at time $t = \pi$. Thus the total time needed to traverse this arc is π seconds.

The average speed is $\dfrac{\text{total distance}}{\text{total time}} \approx \dfrac{7.7253}{\pi}$.

(4) AB

(a) The distance between the particles at time $t = 0$ is given by $|p(0)-q(0)| = |0-(-3)| = 3$

The distance between the particles at any time t is given by $|p(t)-q(t)|$.

The average distance on $0 \le t \le 4$ is given by $\frac{1}{4}\int_0^4 |p(t)-q(t)|\,dt$.

(b) $q(t) = t^3 - 2t^2 + t - 3 \rightarrow q'(t) = 3t^2 - 4t + 1 = (3t-1)(t-1)$.

On $0 \le t \le 3$, particle Q is moving left when $v_q(t) = q'(t) < 0 \rightarrow \frac{1}{3} < t < 1$.

(c) On $0 \le t \le 3$, particle Q is moving right when $v_q(t) = q'(t) > 0 \rightarrow 0 < t < \frac{1}{3}$ or $1 < t < 3$.

$v_p(t) = p'(t) = \frac{3\pi}{2}\cos\left(\frac{\pi}{2}t\right) \rightarrow v_p(t) > 0$ for $0 < t < 1$ and $v_p(t) < 0$ for $1 < t < 3$.

Particles Q and P move in the same direction when $q'(t)$ and $p'(t)$ have the same sign. This occurs when $0 < t < \frac{1}{3}$.

(d) $v_q(2) = 3(2)^2 - 4(2) + 1 > 0$ and $a_q(2) = 6(2) - 4 > 0$. $\leftarrow$ this statement is sufficient justification

Since $v_q(2) \times a_q(2) > 0$, the motion of particle Q is speeding up at $t = 2$.

(e) $v_p(t) = p'(t) = \frac{3\pi}{2}\cos\left(\frac{\pi}{2}t\right)$. Total distance traveled on $0 < t < 2$ is given by $\int_0^2 |v_p(t)|\,dt$.

$$\int_0^2 |v_p(t)|\,dt = \int_0^1 \frac{3\pi}{2}\cos\left(\frac{\pi}{2}t\right)dt - \int_1^2 \frac{3\pi}{2}\cos\left(\frac{\pi}{2}t\right)dt = 3\sin\left(\frac{\pi}{2}t\right)\Big|_0^1 - 3\sin\left(\frac{\pi}{2}t\right)\Big|_1^2 = 6$$

(5)

(a) $g(1) = \int_0^1 e^t\, dt = e^1 - e^0 =$ DO NOT SIMPLIFY !!!

$$g(3) = \int_0^3 f(t)\, dt = \int_0^1 e^t\, dt + \int_1^3 f(t)\, dt = e^1 - e^0 + \frac{1}{2} \times 2 \times e = \text{DO NOT SIMPLIFY !!!}$$

(b) We are given that $\int_{-\sqrt{3}}^{0} f(t)\, dt = -\frac{\sqrt{3}}{2} + \frac{2\pi}{3}$

$$g\left(-\sqrt{3}\right) = \int_0^{-\sqrt{3}} f(t) dt = -\int_{-\sqrt{3}}^{0} f(t)\, dt = -\left(-\frac{\sqrt{3}}{2} + \frac{2\pi}{3}\right) = \text{DO NOT SIMPLIFY !!!}$$

(c) $g'(x) = f(x) = 0$ where $x = -\sqrt{3}$ (we do not have to consider $g'(x) = f(x)$ DNE because f is continuous, hence defined everywhere).

$g'(x) = f(x)$ changes sign from $-$ to $+$ at $x = -\sqrt{3}$ indicating a minimum value. Because $f(x) < 0$ for $-2 < x < -\sqrt{3}$ and $f(x) > 0$ for $-\sqrt{3} < x < 3$, this is an absolute minimum.

(d) Average rate of change is $\frac{f(3) - f(0)}{3 - 0} = -\frac{1}{3}$. For the MVT to apply, f must be differentiable at all points in the interval (0, 3). However, f is not differentiable at $x = 1$.

(e) $g''(2) = f'(2) =$ slope of f at the point where $x = 2 \rightarrow g''(2) = \frac{e - 0}{1 - 3} =$ DO NOT SIMPLIFY !!!

(f) The graph of g has a point of inflection where $g''(x) = f'(x) =$ slope of f changes sign. This occurs at the point where $x = 1$.

(AP Exam grading and "correct" calculus note regarding part (f): all that is required for there to be a point of inflection is a change in sign of the second derivative. Unlike justifying a relative max or min, whether the change is from + to – vs. – to + is irrelevant).