

My Favorite Algebra Problems

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Roots: Basic Facts

Viète's Relations. Given $x^3 + ax^2 + bx + c = 0$ where r , s , and t are the roots, then

$$r + s + t = -a$$

$$rs + rt + st = b$$

$$rst = -c$$

Roots: Basic Facts

Easily generalized for $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ with roots $r_1, r_2, \dots, r_n$.

$$\sum_{i=1}^n r_i = -\frac{a_{n-1}}{a_n}$$

$$\sum_{i,j=1, i \neq j}^n r_i r_j = \frac{a_{n-2}}{a_n}$$

$$\prod_{i=1}^n r_i = (-1)^n \frac{a_0}{a_n}$$

$$\sum_{i=1}^n r_i^2 = \frac{a_{n-1}^2 - 2a_{n-2}a_n}{a_n^2}$$

Roots: Problem 1

If r and s are the roots of $x^2 - 21x + 10 = 0$, then find $r^2 + s^2$.

Clearly, $r + s = 21$ and $rs = 10$. Since

$$(r + s)^2 = r^2 + 2rs + s^2$$

we have

$$21^2 = r^2 + 2(10) + s^2,$$

or

$$r^2 + s^2 = 441 - 20 = 421.$$

Roots: Problem 2

One of the three roots of the equation $x^3 + ax^2 + 21x + 6 = 0$ is the reciprocal of a second root. Find the exact value of a .

[GCTM State Tournament, 2006]

Let the three roots be denoted p , $\frac{1}{p}$, and q . Then the product of the roots is $p \cdot \frac{1}{p} \cdot q = q = -6$. The sum of the pairwise products of the roots is $p \cdot \frac{1}{p} + pq + q \cdot \frac{1}{p} = 1 + pq + \frac{q}{p} = 21$, or $p + p^2q + q = 21p$. Since $q = -6$, this equation becomes $6p^2 + 20p + 6 = 0$, from which we have $p = -3$ or $p = -\frac{1}{3}$. Thus, the roots are -3 , $-\frac{1}{3}$, and -6 . Therefore, a , the sum of the roots, is $-\frac{28}{3}$.

Polynomials: Basic Facts

The Factor Theorem. If $P(a) = 0$, then $x - a$ is a factor of $P(x)$.

The Remainder Theorem. If a polynomial $P(x)$ is divided by $x - a$, then the remainder is the value of $P(a)$.

The Sum of the Coefficients. The value $P(1)$ is the sum of the coefficients of $P(x)$.

Use the roots!

Polynomials: Problem 1

Steve picks a fourth degree polynomial p with nonnegative integer coefficients and challenges Jack to discover the five coefficients. Steve lets Jack pick only two values of x to help him discover the coefficients. Jack picks $x = 1$ and $x = 10$. Steve tells him $p(1) = 9$ and $p(10) = 32,013$. Now Jack knows precisely what the polynomial's coefficients are, and as proof, Jack tells Steve the value of $p(3)$. What is the value of $p(3)$?

[GCTM State Tournament, 2007]

Since $p(1) = 9$, and this is the sum of the coefficients, Jack knows that 9 is at least as big as any of p 's coefficients—in other words, the coefficients are single digits. Jack asks for $p(10)$, which gives the base-10 representation of $p(10)$. This uniquely describes the coefficients of p :
 $p(x) = 3x^4 + 2x^3 + x + 3$. Hence, $p(3) = 303$.

Polynomials: Problem 2

The polynomial $P(x)$ is cubic. What is the largest value of k for which the polynomials $Q_1(x) = x^2 + (k - 29)x - k$ and $Q_2(x) = 2x^2 + (2k - 43)x + k$ are both factors of $P(x)$?

[AIME, 2007]

Because $P(x)$ has three roots, and if $Q_1(x)$ and $Q_2(x)$ are both factors of P , then they must have a common root r . Then $Q_1(r) + Q_2(r) = 0$, and

$$mQ_1(r) + nQ_2(r) = 0$$

for any values of m and n . Choosing $m = 2$ and $n = -1$ gives $15r + 3k = 0$, or $r = -k/5$. Thus,

$$Q_1(r) = \frac{k^2}{25} - (k - 29)\frac{k}{5} - k = 0$$

which is equivalent to $4k^2 - 120k = 0$. The larger root is $k = 30$.

[$k = 30$ implies $Q_1(x) = (x + 6)(x - 5)$, $Q_2(x) = (2x + 5)(x + 6)$, and $P(x) = (x + 6)(x - 5)(2x + 5)$.]

Polynomials: Problem 3

If $f(x) = 3x^2 - 2x + 5$ and $f(g(x)) = 12x^4 + 56x^2 + 70$, then compute the product of all possible values of the sum of the coefficients of $g(x)$.

[ARML Practice, 2007]

The sum of the coefficients of $g(x)$ is $g(1)$.

Since $f(g(x)) = 3(g(x))^2 - 2g(x) + 5$, we have

$$f(g(1)) = 3(g(1))^2 - 2g(1) + 5 = 12 + 56 + 70,$$

or $3(g(1))^2 - 2g(1) - 133 = 0$. We factor to obtain

$$[3g(1) + 19][g(1) - 7] = 0$$

so that $g(1) = 7$ or $g(1) = -19/3$. The product is $-133/3$.

Telescoping Sums: Basic Facts

A *telescoping sum* is a sum all of whose terms cancel except the first and last.

Example. The sum

$$\sum_{n=1}^k \frac{1}{n(n+1)}$$

can be written equivalently as

$$\sum_{n=1}^k \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

Then we see that the “middle” terms cancel:

$$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots - \frac{1}{k} + \frac{1}{k} - \frac{1}{k+1} = 1 - \frac{1}{k+1}.$$

Rewrite the given expression.

Telescoping Sums: Problem 1

In terms of n , evaluate

$$\sum_{k=1}^n \frac{k}{(k+1)!}.$$

We write

$$\frac{k}{(k+1)!} = \frac{k+1-1}{(k+1)!} = \frac{k+1}{(k+1)!} - \frac{1}{(k+1)!} = \frac{1}{k!} - \frac{1}{(k+1)!}.$$

Then the sum becomes

$$\left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \cdots + \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) = 1 - \frac{1}{(n+1)!}$$

Telescoping Sums: Problem 2

A sequence is defined as follows: $a_1 = a_2 = a_3 = 1$, and, for all positive integers n ,

$$a_{n+3} = a_{n+2} + a_{n+1} + a_n.$$

Given that $a_{28} = 6090307$, $a_{29} = 11201821$, and $a_{30} = 20603361$, find the remainder when

$$\sum_{k=1}^{28} a_k$$

is divided by 1000.

[AIME II, 2006]

Telescoping Sums: Problem 2

Because $a_{k+3} - a_{k+2} = a_{k+1} + a_k$ for all positive integers k , we have

$$\sum_{k=1}^n (a_{k+3} - a_{k+2}) = \sum_{k=1}^n (a_{k+1} + a_k)$$

Let $S_n = \sum_{k=1}^n a_k$. Notice that $\sum_{k=1}^n (a_{k+3} - a_{k+2})$ is telescoping, and becomes simply $a_{n+3} - a_3$. Also,

$$\sum_{k=1}^n (a_{k+1} + a_k) = (S_n - a_1 + a_{n+1}) + S_n.$$

Thus $a_{n+3} - a_3 = S_n - a_1 + a_{n+1} + S_n$, so that

$$S_n = \frac{1}{2}(a_{n+3} - a_{n+1}) = \frac{1}{2}(a_{n+2} + a_n).$$

In particular, $S_{28} = \frac{1}{2}(a_{30} + a_{28}) = \frac{1}{2}(20603361 + 6090307) = 13346834$ so the remainder when divided by 1000 is 834.

Arithmetic-Geometric Sums: Definitions

A sum such as the following

$$\frac{3}{4} + \frac{7}{16} + \frac{11}{64} + \cdots + \frac{3 + 4k}{4^{k-1}} + \cdots$$

has a numerator that is arithmetic and a denominator that is geometric.

To evaluate, we set the sum equal to S and compute $S - \frac{1}{4}S$.

This sum is either *geometric* or includes the original series.

Arithmetic-Geometric Sums: Problem 1

Evaluate $\sum_{n=1}^{\infty} \frac{F_n}{10^{n+1}}$, where F_n is the n th Fibonacci number, i.e.,
 $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3$, etc.
[GCTM State Tournament, 2005]

Arithmetic-Geometric Sums: Problem 1

Call the sum S . Then

$$S = \frac{1}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \cdots$$

and

$$\frac{1}{10}S = \frac{1}{10^3} + \frac{1}{10^4} + \frac{2}{10^5} + \frac{3}{10^6} + \cdots$$

Then

$$S - \frac{1}{10}S = \frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^5} + \frac{2}{10^6} + \cdots = \frac{1}{100} + \frac{1}{100}S,$$

or

$$\frac{9}{10}S = \frac{1}{100}(S + 1),$$

which implies $90S = S + 1$ whose solution is $S = \frac{1}{89}$.

Two Sums and a Product

Let a and r be positive real numbers. If $\sum_{n=1}^{10} ar^n = 18$ and $\sum_{n=1}^{10} \frac{1}{ar^n} = 6$, then

find $\prod_{n=1}^{10} ar^n$.

[ARML Practice, 2007]

First, note that $\prod_{n=1}^{10} ar^n = a^{10} r^{55}$. The second sum can be written as

$$6 = \frac{1}{ar} + \frac{1}{ar^2} + \cdots + \frac{1}{ar^{10}} = \frac{r^9 + \cdots + r + 1}{ar^{10}} = \frac{18/(ar)}{ar^{10}} = \frac{18}{a^2 r^{11}}$$

Hence, $a^2 r^{11} = 18/6 = 3$. Thus,

$$a^{10} r^{55} = (a^2 r^{11})^5 = 3^5 = 243.$$

A Number and Its Reciprocal

If $x + \frac{1}{x} = 3$, then find $x^2 + \frac{1}{x^2}$.

Squaring the given equation, we have

$$x^2 + 2 + \frac{1}{x^2} = 9,$$

so then

$$x^2 + \frac{1}{x^2} = 7.$$

Fun with Coefficients

Find the sum of the coefficients obtained after expanding the product

$$(1 - 3x + 3x^2)^{75}(1 + 5x - 5x^2)^{125}.$$

Let $x = 1$ to get $(1)^{75}(1)^{125} = 1$.

The Law of Cosines

The sides of a triangle are x , y , and $\sqrt{x^2 + xy + y^2}$. Find the measure of the largest angle.

[GCTM State Tournament 2006]

The largest angle is opposite the largest side; the largest side is $\sqrt{x^2 + xy + y^2}$. By the Law of Cosines, we have

$$\left(\sqrt{x^2 + xy + y^2}\right)^2 = x^2 + y^2 - 2xy \cos \theta$$

$$x^2 + xy + y^2 = x^2 + y^2 - 2xy \cos \theta$$

$$xy = -2xy \cos \theta$$

$$-\frac{1}{2} = \cos \theta$$

$$\theta = 120^\circ$$

A System of Equations

Consider positive integers A , B , and C such that $A^2 + B - C = 100$ and $A + B^2 - C = 124$. Find the value of C .

[GCTM State Tournament 2006]

Subtract the first equation from the second and factor to obtain

$$(B - A)(A + B - 1) = 24.$$

Since the sum of these two factors is $2B - 1$ (which is odd), then one factor is odd and the other even. Thus, we need only consider factors of 24 where one factor is odd and the other even; the only choices are 3, 8 and 1, 24. Hence, we have one of two systems.

$$\begin{cases} B - A = 3 \\ A + B - 1 = 8 \end{cases} \quad \begin{cases} B - A = 1 \\ A + B - 1 = 24 \end{cases}$$

The first system gives $A = 3$, $B = 6$, $C = -85$ which we rule out since C is not positive. The second system gives $A = 12$, $B = 13$, $C = 57$.

Nested Radicals

Evaluate

$$\sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}}$$

Let $x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}}$. Then upon squaring both sides, we get

$$x^2 = 6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}$$

$$x^2 = 6 + x$$

$$x^2 - x - 6 = 0$$

$$x = 3$$

Summary of Basic Techniques

Roots Use Viète's Relations – only find the roots as a last resort

Polynomials Factors, remainders, $f(1)$

Telescoping Series Rewrite the expression

Arithmetic-Geometric Turn it into a telescoping sum