LEONHARD EULER

A (Very) Brief Look at His Life and Mathematics

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Outline

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1. Euler's Life
   - Early Years
   - St. Petersberg
   - Berlin
   - Back in St. Petersberg
2. Euler's Mathematics
   - Euler's Formula
   - Sum of Series
   - Integers and Primes
3. References

Euler's Early Years

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- Johann Bernoulli (1667-1748)

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"I had to register in the faculty of theology, and I was to apply myself ... to the Greek and Hebrew languages, but not much progress was made, for I turned most of my time to mathematical studies, and by my happy fortune the Saturday visits to Johann Bernoulli continued."
—*Euler, quoted from Truesdell's preface to Elements of Algebra*

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- Daniel invited Euler to join him
- Euler arrived in 1727 as professor of physics

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Euler in St. Petersberg

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- Solved the "Basel Problem"

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- Head of the Academy was stifling
- Deteriorating eyesight, begun in 1738

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Discoveries during this time:

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- Number theory and inaugurated analytic number theory
- Landmark physics book, *Mechanica*
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Offered a position by Prussia's Frederick the Great in 1741

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- *Introduction to Infinite Analysis* and a text on differential calculus
- *Letters to a German Princess*

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These factors led Euler to go back to St. Petersberg in 1766

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St. Petersberg, Phase Two

- By 1771, Euler was totally blind
- Wife died in 1773
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- None of these events stopped the mathematics!

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- *Opera Omnia*
  - Begun in 1911, still going
  - 86 Volumes planned, 73 published
  - Total of 25,000 pages

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Euler's Conception of Series

A Series is a Polynomial with Infinitely Many Terms

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$$

Converges for $-1 < x < 1$; diverges otherwise.

Euler's Conception of Series

Euler considered the following:

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - + \cdots$$

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Euler thought

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - + \cdots = \frac{1}{2}$$

Euler's Formula

Euler begins with the following relations:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \cdots ,$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots ,$$

and

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .$$

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Euler decides that *imaginary values* also make the series representation true!

Euler's Formula

He replaces x with ix...

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

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e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots \\
&= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} - + \cdots \\
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&= \cos x + i \sin x
\end{aligned}
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Hence, $e^{ix} = \cos x + i \sin x$.

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Sum of the Reciprocals of Integers

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Appears to be infinite, but gets there slowly.

Sum of the Reciprocals of Integers

Euler uses

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

to start.

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$$\log 0 = -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right)$$

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$$\sum_{k=1}^{\infty} \frac{1}{k} = -\log 0$$

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$$\log 0 = -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right)$$

$$\sum_{k=1}^{\infty} \frac{1}{k} = -\log 0 = \log 0^{-1} = \log \frac{1}{0}$$

Sum of the Reciprocals of Integers

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to start. Let $x = 1$. Then

$$\log 0 = -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right)$$

$$\sum_{k=1}^{\infty} \frac{1}{k} = -\log 0 = \log 0^{-1} = \log \frac{1}{0} = \log \infty = \infty.$$

Sum of the Reciprocals of Integers

$$\sum_{k=1}^{n} \frac{1}{k} = \log_e(n+1) + 0.577218\ldots$$

Sum of the Reciprocals of Integers

$$\sum_{k=1}^{n} \frac{1}{k} = \log_e(n+1) + 0.577218\ldots$$

Euler's Constant $\gamma \approx 0.577218$

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Brute force:

$$\sum_{k=1}^{10} \frac{1}{k^2} \approx 1.54977, \qquad \sum_{k=1}^{100} \frac{1}{k^2} \approx 1.63498,$$

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Brute force:

$$\sum_{k=1}^{10} \frac{1}{k^2} \approx 1.54977, \qquad \sum_{k=1}^{100} \frac{1}{k^2} \approx 1.63498, \qquad \sum_{k=1}^{1000} \frac{1}{k^2} \approx 1.64393$$

The Basel Problem

Is $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$ finite or not?
Brute force:

$$\sum_{k=1}^{10} \frac{1}{k^2} \approx 1.54977, \qquad \sum_{k=1}^{100} \frac{1}{k^2} \approx 1.63498, \qquad \sum_{k=1}^{1000} \frac{1}{k^2} \approx 1.64393$$

Appears to be finite, so what is the exact value?

The Basel Problem

If $P(x) = 0$ is a polynomial of degree n and $P(0) = 1$, then

$$P(x) = \left(1 - \frac{x}{r_1}\right)\left(1 - \frac{x}{r_2}\right)\cdots\left(1 - \frac{x}{r_n}\right)$$

where the r's are the roots.

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$$P(x) = \left(1 - \frac{x}{r_1}\right)\left(1 - \frac{x}{r_2}\right)\cdots\left(1 - \frac{x}{r_n}\right)$$

where the r's are the roots. Consider

$$P(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + - \cdots$$

(In other words, let P be an infinite polynomial).

The Basel Problem

Then

$$P(x) = x\left(\frac{1-\dfrac{x^2}{3!}+\dfrac{x^4}{5!}-\dfrac{x^6}{7!}+-\cdots}{x}\right)$$

The Basel Problem

Then

$$\begin{aligned} P(x) &= x\left(\frac{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + - \cdots}{x}\right) \\ &= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots}{x} \end{aligned}$$

The Basel Problem

Then

$$P(x) = x\left(\frac{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+-\cdots}{x}\right)$$
$$= \frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+-\cdots}{x}$$
$$= \frac{\sin x}{x}$$

The Basel Problem

Then

$$\begin{aligned} P(x) &= x\left(\frac{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+-\cdots}{x}\right) \\ &= \frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+-\cdots}{x} \\ &= \frac{\sin x}{x} \end{aligned}$$

So the roots of $P(x)$ are roots of $\sin x$.

The Basel Problem

$$P(x) = \left(1 - \frac{x}{\pi}\right)\left(1 - \frac{x}{-\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 - \frac{x}{-2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\cdots$$

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$$\begin{aligned} P(x) &= \left(1-\frac{x}{\pi}\right)\left(1-\frac{x}{-\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1-\frac{x}{-2\pi}\right)\left(1-\frac{x}{3\pi}\right)\cdots \\ &= \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots \end{aligned}$$

since $\pm k\pi$ are the roots of $\sin x$.

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since $\pm k\pi$ are the roots of $\sin x$. Expand:

$$1-\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\cdots\right)x^2+\cdots$$

The Basel Problem

Two ways to represent P:

$$P(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + - \cdots$$

and

$$P(x) = 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots \right) x^2 + \cdots$$

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Equate coefficients of x^2 to get

$$-\frac{1}{3!} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots$$

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$$\begin{aligned} -\frac{1}{3!} &= \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots \\ &= -\frac{1}{\pi^2} \left(1 + \frac{1}{4} + \frac{1}{9} + \cdots \right) \end{aligned}$$

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The Basel Problem

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Outline

Sums of Integers, Products of Primes

Let $x = \sum_{k=1}^{\infty} \frac{1}{k}$.

Then

$$\frac{1}{2}x = x - \frac{1}{2}x$$

Sums of Integers, Products of Primes

Let $x = \sum_{k=1}^{\infty} \frac{1}{k}$.

Then

$$\begin{aligned} \frac{1}{2}x &= x - \frac{1}{2}x \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots \end{aligned}$$

Sums of Integers, Products of Primes

Let $x = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$.

Then

$$\begin{aligned}
\frac{1}{2}x &= x - \frac{1}{2}x \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots \\
&= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots
\end{aligned}$$

Sums of Integers, Products of Primes

Continue:

$$\frac{1}{3}x = \frac{1 \cdot 2}{2 \cdot 3}x \quad = \frac{1}{2}x - \frac{1}{3}\left(\frac{1}{2}x\right)$$

Sums of Integers, Products of Primes

Continue:

$$\begin{aligned}\frac{1}{3}x = \frac{1 \cdot 2}{2 \cdot 3}x \quad &= \frac{1}{2}x - \frac{1}{3}\left(\frac{1}{2}x\right) \\ &= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots - \frac{1}{3} - \frac{1}{9} - \frac{1}{15} - \cdots\end{aligned}$$

Sums of Integers, Products of Primes

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$$\begin{aligned}\frac{1}{3}x = \frac{1\cdot 2}{2\cdot 3}x \quad &= \frac{1}{2}x - \frac{1}{3}\left(\frac{1}{2}x\right) \\ &= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots - \frac{1}{3} - \frac{1}{9} - \frac{1}{15} - \cdots \\ &= 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots\end{aligned}$$

Sums of Integers, Products of Primes

Continue:

$$\begin{aligned}
\frac{1}{3}x = \frac{1\cdot 2}{2\cdot 3}x \quad &= \frac{1}{2}x - \frac{1}{3}\left(\frac{1}{2}x\right) \\
&= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots - \frac{1}{3} - \frac{1}{9} - \frac{1}{15} - \cdots \\
&= 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots
\end{aligned}$$

$$\frac{1\cdot 2\cdot 4}{2\cdot 3\cdot 5}x \quad = \frac{1}{3}x - \frac{1}{5}\left(\frac{1}{3}x\right)$$

Sums of Integers, Products of Primes

Continue:

$$\begin{aligned}
\frac{1}{3}x = \frac{1\cdot 2}{2\cdot 3}x \quad &= \frac{1}{2}x - \frac{1}{3}\left(\frac{1}{2}x\right) \\
&= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots - \frac{1}{3} - \frac{1}{9} - \frac{1}{15} - \cdots \\
&= 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots \\
\frac{1\cdot 2\cdot 4}{2\cdot 3\cdot 5}x \quad &= \frac{1}{3}x - \frac{1}{5}\left(\frac{1}{3}x\right) \\
&= 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots - \frac{1}{5} - \frac{1}{25} - \frac{1}{35} - \cdots
\end{aligned}$$

Sums of Integers, Products of Primes

Continue:

$$\begin{aligned}
\frac{1}{3}x = \frac{1\cdot 2}{2\cdot 3}x &= \frac{1}{2}x - \frac{1}{3}\left(\frac{1}{2}x\right) \\
&= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots - \frac{1}{3} - \frac{1}{9} - \frac{1}{15} - \cdots \\
&= 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots
\end{aligned}$$

$$\begin{aligned}
\frac{1\cdot 2\cdot 4}{2\cdot 3\cdot 5}x &= \frac{1}{3}x - \frac{1}{5}\left(\frac{1}{3}x\right) \\
&= 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots - \frac{1}{5} - \frac{1}{25} - \frac{1}{35} - \cdots \\
&= 1 + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots
\end{aligned}$$

Sums of Integers, Products of Primes

Thus,

$$\frac{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 \cdot \cdots}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot \cdots} x = 1$$

Sums of Integers, Products of Primes

Thus,

$$\frac{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 \cdot \cdots}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot \cdots} x = 1$$

Or,

$$x = \frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot \cdots}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 \cdot \cdots}$$

Sums of Integers, Products of Primes

Thus,

$$\frac{1\cdot 2\cdot 4\cdot 6\cdot 10\cdot 12\cdot\dots}{2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot\dots}x = 1$$

Or,

$$x = \frac{2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot\dots}{1\cdot 2\cdot 4\cdot 6\cdot 10\cdot 12\cdot\dots}$$

Hence,

$$\sum_{k=1}^{\infty}\frac{1}{k} = \prod_{p=2}^{\infty}\frac{p}{p-1}$$

References

"Read Euler, read Euler. He is the master of us all." —*Laplace*

- Dr. Euler's Fabulous Formula, by Paul Nahin
- Euler: The Master of Us All by William Dunham
- Euler Through Time: A New Look at Old Themes by V. S. Varadarajan