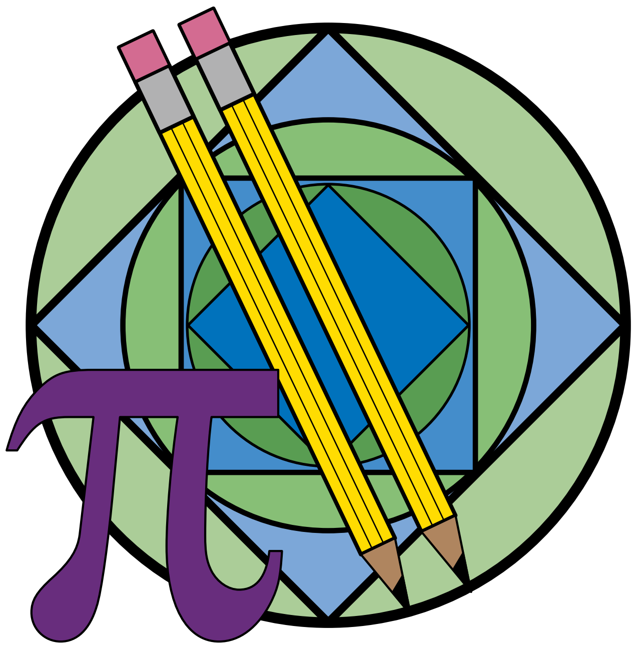
Mastering Competition Mathematics
Volume 2: Probability and Number Theory



Chuck Garner

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Mastering Competition Mathematics
Volume 2: Probability and Number Theory

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by Dr. Charles R. Garner, Jr
Rockdale Magnet School for Science and Technology, Conyers, Georgia USA

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PREFACE

> I'd say that about 82% of what I write is bad, but don't go by me; I'm as bad a judge as I am a writer. Look, if it were all good, you'd be paying twice as much for this book. So relax, read it, and if you don't enjoy it, remember that you're saving money.
>
> — *George Burns*

The purpose of this book is provide high school students with practice solving typical math competition problems as found on actual high school competitions. All the problems contained in this book are sourced from mathematics tournaments which have occurred in the state of Georgia, USA.

Each chapter begins with some expository material with examples. The expository material attempts to explain to the reader some basic and important mathematical problem-solving techniques, illustrated by a variety of examples. Following some examples are exercises. The exercises give the reader a chance to test their understanding immediately. The exercises are numbered consecutively with the problems. The examples and exercises are competition problems, but specifically curated to help the reader understand a topic they may be learning for the first time.

The problem sections in each chapter are divided into two parts, "JV" (for "Junior Varsity") and "Varsity". In Georgia, nearly every tournament is offered in these divisions. The JV division is typically for students who have not yet enrolled in a precalculus course while Varsity is for any K-12 student. Many JV problems are easier than the Varsity problems, but that does not mean that every JV problem is easy! I have attempted to put the JV problems and Varsity problems in order of difficulty, but that is always a subjective measure.

The problems are divided into JV and Varsity to provide mathematical challenges for students as they move through high school. Middle and high school students using this book can use the JV problems to master the topic while in middle school or the early years of high school. Then the student can tackle the Varsity problems afterwards. Even the exposition can and should be re-read. The first reading does not always suffice to master a topic; multiple readings are warranted.

"To reread a book is to read a different book. The reader is different. The meaning is different." — Johnny Rich

As a result of this dual purpose to support younger and older students, there are many problems where the same concept appears. Usually, problem books eschew too many of the same type of problem in order to expose the student to a wide variety of problem-solving techniques. This is necessary if the goal is to prepare the student for, say, the AIME or the USAMO. The goal of this series of books is more modest: the student should become so familiar with basic problem-solving techniques, mathematical properties, and important formulæ that they become second-nature. That is nearly impossible without repetition. Hence, you will find the same *type* of problem repeated often.

But not literally the same problem!

The repetition of problems also amply demonstrates what competition test-writers consider good topics to put on their tests. This gives the student a glimpse into what many test-writers either would like the competitor to know how to solve, or what they think will stump the competitor.

Full solutions to the all the exercises and and all the problems are included in the latter half of the book. A list of all the answers appears just before the index. This list makes it easier to check your answer without looking at the solution.

In putting together this book, I have been amazed at the ingenuity of the problem creators. This is especially evident with the problems originating from the high school tournament hosts, as opposed to the university hosts. The mathematically adept students in high school generally write problems to stymie their competition, and it shows! Many of the problems from Walton High School, Milton High School, Columbus High School, and the North Fulton competitions (where the problems were written by the students at those schools) are some of the most difficult problems in this book. That is not to say that the problems from other tournaments are always easy!

This is the second of four volumes designed to help the reader "master competition mathematics." This volume focuses on combinatorics, probability, and number theory. These are broad categories, and as such, there is a wide variety of problems here among the 1209 problems in this volume.

Finally, I would like to thank Precious Babajide, who was my student for her junior and senior years of high school (2023-2025). She proofread every page of this book. She caught a huge number of typos, many spelling errors, and some mathematical errors! Precious also suggested improvements on the wording of some of the solutions.

Chuck Garner
May 2025

2 NUMBER THEORY

Why are numbers beautiful? It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is.
— *Paul Erdős*

Number theory is simply the study of properties of integers. One may think that the integers are fairly boring, but they hold some fascinating and surprising properties. Along with algebra, geometry, and combinatorics, number theory constitutes one of the major branches of competition mathematics. Number theory has become such a large part of competition mathematics not least for the interest it stimulates, but also for the fact that many students are able to solve many problems without a great deal of advanced knowledge of the subject. However, some advanced knowledge of the subject is certainly useful and it is most definitely interesting!

Digits

It is useful sometimes to represent an integer in terms of its digits. For instance, a three-digit integer N has a digit in the hundreds place, the tens place, and the units place. If these digits are h, t, and u, then N can be expressed as $N = 100h + 10t + u$. We use this in our first example.

Example 1. Suppose M is an integer equal to 10 times the sum of its digits and N is an integer equal to 5 times the sum of its digits. Find the maximum possible value of $M + N$. [CHS]

(A) 75 **(B)** 115 **(C)** 135 **(D)** 185 **(E)** 205

The largest four-digit integer is 9999 whose digit sum is only 36.

Solution. First, we determine the number of digits M has. It cannot have four or more digits since an integer greater than or equal to 1000 is too large to be 10 times the sum of the digits. Suppose M has three digits. Then $M = 100h + 10t + u = 10(h + t + u)$. This implies $100h + u = 10h + 10u$ so that $90h = 9u$. But since h and u are digits, the only solution is $h = u = 0$ which cannot give a three-digit number. Hence, M is a two-digit integer and we can write $M = 10t + u = 10(t + u)$ so that $u = 10u$. This implies $u = 0$ and h can be any digit. Since we want the largest, we choose $t = 9$. Thus, $M = 90$.

By similar reasoning, N must also be a two-digit integer. From $N = 10t + u = 5(t+u)$ we obtain $5t = 4u$. Since t and u must be digits, the only solution is $t = 4$ and $u = 5$, implying $N = 45$. Finally, $M + N = 90 + 45 = 135$. Thus, the answer is **C**. ◆

Example 2. If each distinct letter in $\sqrt{GACS} = GS$ represents a distinct positive digit in base 10, compute the sum $G + A + C + S$.

[GAC 2003, Comprehensive]

Solution. From $\sqrt{GACS} = GS$, we can write $GACS = (GS)^2 = (10G + S)^2$. The smallest two-digit integer that results in a four-digit integer when squared is 32: $32^2 = 1024$. So $32 \leq GS \leq 99$. Now, since $GACS = (GS)^2$, GS and its square each have leading digit G. However, for $3 \leq G \leq 9$, the square of none of these digits has a leading digit of G. Hence, there must be a carry involved in computing $(GS)^2$. The largest carry possible is 1, so we need a digit whose square has leading digit one less than the digit. The only one that works is $G = 9$ ($G^2 = 81$).

Now we examine the units digit. The units digit of $(10G + S)^2$ results from S^2, and this must be the same units digit as that of $GACS$, which is S. This narrows the possibilities for S to 0, 1, 5, or 6, since the square of these digits have a units digit of 0, 1, 5, and 6. Hence, we have 90, 91, 95, and 96 as the possibilities for GS. However, the squares of 90 and 91 do not give us the leading digit 9. Hence, it must be one of 95 and 96. Checking, $95^2 = 9025$ and $96^2 = 9216$. We can rule out 95^2 since the problem told us that the digits are positive. So 9216 is $GACS$ and $G + A + C + S = 9 + 2 + 1 + 6 = 18$. Thus, the answer is **18**. ◆

Exercises

1. The digits 1, 2, 3, 4, 5, 6, 7, and 9 are used to form four 2-digit prime numbers, with each digit used exactly once. What is the sum of these four primes? [Cobb 2005, Varsity]

   (A) 150 **(B)** 165 **(C)** 180 **(D)** 190 **(E)** 205

2. If A, B, C, and D are distinct digits and $ABCD \cdot 9 = DCBA$, then what is $A + B + C + D$? [North Fulton 2012, Varsity]

   (A) 12 **(B)** 14 **(C)** 16 **(D)** 18 **(E)** None of these

3. There are 120 five-digit numbers that can be formed by permuting the digits 1, 2, 3, 4, and 5. The sum of all 120 such numbers is [GSW 2002]

   (A) 2876540 **(B)** 3999960 **(C)** 4969960 **(D)** 5600610
   (E) None of these

Divisibility

We will begin with the defintion of divisibility.

Definition. An integer n is divisible by a positive integer d if there is some integer k such that $n = kd$. We can write that n is divisible by d with the notation $d \mid n$.

We read $d \mid n$ as "d divides n".

For instance $12 \mid 48$ since there is an integer, namely 4, such that $48 = 4 \cdot 12$. We can say that 12 does not divide 49 ($12 \nmid 49$) since there is no integer that when multiplied to 12 gives 49.

Notice we do not define divisibility in terms of division, or in terms of "goes into evenly" even though this is implied. We define divisibility in terms of multiplication.

The simplest properties of divisibility are the divisibility rules. The easiest divisibility rule of all is one learned many years ago: $2 \mid n$ if n is even. The next divisibility rule one learns (or figures out) is that if n has a units digit of 5 or 0, then $5 \mid n$. Likewise, $10 \mid n$ if n has a units digit of 0. Others follow relatively quickly.

- $3 \mid n$ if and only if the sum of the digits of n is divisible by 3.
- $4 \mid n$ if and only if the tens digit of n and the units digit of n together make an integer divisible by 4.
- $6 \mid n$ if and only if $2 \mid n$ and $3 \mid n$.
- $8 \mid n$ if and only if the hundreds, tens, and units digits of n together make a number divisible by 8.
- $9 \mid n$ if and only if the sum of its digits of n is divisible by 9.

You no doubt notice that 7 is missing from the list. It is not that 7 does not have a rule, it is just that it is difficult to describe without an example. Consider the integer 39548488. To test divisibility by 7, we group the digits in threes, beginning with the right-hand side, like we are putting the comma separators between every thousand: 39, 548, and 488. Then we take the alternating sum of these new numbers: $39 - 548 + 488 = -21$. Since -21 is divisible by 7, so is the original integer. Why does this work?

Let's consider a simple case, which can be generalized. Consider a nine-digit integer $d_9d_8d_7d_6d_5d_4d_3d_2d_1$ where each d_i is a single digit. This number can be expressed as the sum of three 3-digit numbers multiplied by the appropriate power of 10: $1000000d_9d_8d_7 + 1000d_6d_5d_4 + d_3d_2d_1$. Now,

$$\begin{aligned}
&1000000d_9d_8d_7 + 1000d_6d_5d_4 + d_3d_2d_1 \\
&\qquad = 999999d_9d_8d_7 + d_9d_8d_7 + 1001d_6d_5d_4 - d_6d_5d_4 + d_3d_2d_1 \\
&\qquad = 999999d_9d_8d_7 + 1001d_6d_5d_4 + d_9d_8d_7 - d_6d_5d_4 + d_3d_2d_1 \\
&\qquad = 7(142857d_9d_8d_7 + 143d_6d_5d_4) + d_9d_8d_7 - d_6d_5d_4 + d_3d_2d_1
\end{aligned}$$

and clearly, $7 \mid 7(142857d_9d_8d_7 + 143d_6d_5d_4)$, so we only need to determine whether $d_9d_8d_7 - d_6d_5d_4 + d_3d_2d_1$ is divisible by 7. In practice, using this divisibility rule leaves us with a three-digit number that we then need to actually divide by 7.

Since 999999 and 1001 are both also divisible by 13, the same divisibility test for 7 also works for 13.

The divisibility test for 11 is similar to the test for 7. Consider the integer 51258537. To test for divisibility by 11, we find the alternating sum of the digits:

$5-1+2-5+8-5+3-7=0$. Since 0 is divisible by 11, the original integer is divisible by 11. This works by a similar argument to the test for divisibility for 7.

Consider a five-digit integer $d_5d_4d_3d_2d_1$. We can write this as

$$\begin{aligned}10000d_5 &+ 1000d_4 + 100d_3 + 10d_2 + d_1\\ &= 9999d_5 + d_5 + 1001d_4 - d_4 + 99d_3 + d_3 + 11d_2 - d_2 + d_1\\ &= 9999d_5 + 1001d_4 + 99d_3 + 11d_2 + d_5 - d_4 + d_3 - d_2 + d_1\\ &= 11(909d_5 + 91d_4 + 9d_3 + d_2) + d_5 - d_4 + d_3 - d_2 + d_1\end{aligned}$$

and clearly, $11 \mid 11(909d_5 + 91d_4 + 9d_3 + d_2)$, so we need only determine whether $d_5 - d_4 + d_3 - d_2 + d_1$ is divisible by 11.

These divisibility rules aid in finding prime factors of an integer. One only has to check divisibility by 2, 3, 5, 7, 11, 13, and so on, to find the prime factors. Moreover, factors (not necessarily prime ones) of an integer n occur in pairs where one member of the pair is less than $\sqrt{n}$ and the other is greater than $\sqrt{n}$. In the case where n is a perfect square, then one factor is certainly equal to $\sqrt{n}$.

Example 3. Find the sum of the distinct prime factors of 2008. [AASU 2008]

(A) 234 **(B)** 253 **(C)** 257 **(D)** 259 **(E)** 504

Solution. Clearly, $8 \mid 2008$ since the last three digits form an integer divisible by 8. Thus, $2008 = 8 \cdot 251 = 2^3 \cdot 251$. Now we must determine whether 251 is prime or not; 251 does not meet the divisibility criteria for 3, 5, 7, 11, or 13. Since these are the only possible prime divisors less than $\sqrt{251} < 16$, it follows that 251 must be prime. Hence the prime divisors of 2008 are 2 and 251, and their sum is 253. Thus, the answer is **B**. ◆

Example 4. Which of the following numbers is prime? [Walton 2013, Varsity]

(A) 301 **(B)** 303 **(C)** 305 **(D)** 307 **(E)** 309

Solution. Certainly $5 \mid 305$ and, since the sum of their digits is a multiple of 3, $3 \mid 303$ and $3 \mid 309$. This leaves 301 and 307. Trying divisibility by 7 reveals that $301 = 7 \cdot 43$. Hence, the prime number is 307. Thus, the answer is **D**. ◆

Related to questions concerning factors of an integer are questions concerning factors of the factorial of an integer. Recall that n-factorial is $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$.

Example 5. Determine the largest integer n such that 10^n divides $26!$.

[AASU 2015]

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 10

Solution. To obtain a factor of 10, we need to pair factors of 2 and 5. The number of factors of 2 in 26! is greater than the number of factors of 5. It follows that some factors of 2 cannot be paired with a 5 to create a factor of 10. It remains then to determine how many factors of 5 there are. Every multiple of 5 in the range from 1 to 26 will contribute a factor of 5 in 26!. Using the floor function, we see that there are $\lfloor 26/5 \rfloor = 5$ multiples of 5. However, some

There are more evens in the first 26 integers than there are multiples of 5, you know.

The floor of the real number x is the greatest integer less than or equal to x.

of these multiples of 5 contain another factor of 5; in particular, the number $25 = 5^2$. This contains a second factor of 5. Hence, there are $5 + 1 = 6$ factors of 5 in 26! that will each get paired with a 2 to create a factor of 10. We find that $10^6 \mid 26!$. Thus, the answer is **B**. ◆

Example 6. Find the sum of all possible values of n, $0 < n < 15$, such that $n! + (n+1)! + (n+2)!$ is an integral multiple of 36. [Cobb 2000, Varsity]

(A) 39 **(B)** 43 **(C)** 58 **(D)** 60 **(E)** 94

Solution. First, note that $n! + (n+1)! + (n+2)!$ is factorable.

$$\begin{aligned} n! + (n+1)! + (n+2)! &= n!(1 + n + 1 + (n+1)(n+2)) \\ &= n!(n + 2 + n^2 + 3n + 2) \\ &= n!(n^2 + 4n + 4) \\ &= n!(n+2)^2. \end{aligned}$$

Next, the expression $n!(n+2)^2$ will be divisible by $36 = 2^2 \cdot 3^2$ if $36 \mid n!$ or if $36 \mid (n+2)^2$. We see that the smallest n for which $36 \mid n!$ is $n = 6$ since $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 36 \cdot 20$. Thus, $36 \mid n!$ for all n such that $6 \leq n < 15$. The smallest n for which $36 \mid (n+2)^2$ is $n = 4$ since $(4+2)^2 = 36$. We have not ruled out the values 1, 2, 3, and 5 for n, but plugging these into the expression yields 9, $2 \cdot 16$, $6 \cdot 25$, and $120 \cdot 49$, none of which are divisible by 36. Hence, the sum of the values of n which satisfy the condition is $4 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 = 4 + 9(6 + 14)/2 = 4 + 90 = 94$. Thus, the answer is **E**. ◆

The sum of consecutive integers is equal to the sum of the first and last integers, multiplied by the number of integers, divided by 2.

Exercises

4. The sum of the distinct prime factors of 2007 is [AASU 2007]

   (A) 59 **(B)** 183 **(C)** 226 **(D)** 545 **(E)** 1002

5. The pattern of digits in x^2 is $AABB$. Determine the value of $A + B$. [Cobb 2003, Varsity]

   (A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

6. What is the largest power of 5 that divides 125!? [AASU 1998]

   (A) 5^5 **(B)** 5^{11} **(C)** 5^{25} **(D)** 5^{31} **(E)** 5^{35}

7. When 616! is multiplied out, how many zeros appear on the end?

   (A) 61 **(B)** 62 **(C)** 123 **(D)** 151 **(E)** 302

Another type of problem in which factors and divisibility is important is in solving an equation in integers. Consider the equation $mn = 4(m+n)+2$ where we want all solutions where m and n are integers. This equation is approached by a factoring method called "*Simon's Favorite Factoring Trick*" or simply "SFFT". It goes like this. First, distribute and then set the equation equal to the constant: $mn - 4m - 4n = 2$. Now we take the coefficient of the linear terms and square it, then add this to both sides: $mn - 4m - 4n + 16 = 18$. The result is that the left-hand side of the equation is now factorable: $(m-4)(n-4) = 18$. At this point, we use the fact that m and n are integers to realize that $m-4$ and $n-4$ are integers whose product is 18. There are only so many possibilities: 1 and 18, 2 and 9, 3 and 6, -1 and -18, -2 and -9, or -3 and -6. Each pair of factors must then be equal to one of $m-4$ and $n-4$. This results in the equations below.

Simon Rubenstein-Salzedo really liked this trick, and on an `artofproblemsolving.com` *message board in 2003, jokingly named it after himself! But the name stuck!*

$$\begin{aligned} m-4 &= 1, \quad 2, \quad 3, \quad -1, \quad -2, \quad -3 \\ n-4 &= 18, \quad 9, \quad 6, \quad -18, \quad -9, \quad -6 \end{aligned}$$

Adding four to each of the values gives the solutions

$$\begin{aligned} m &= 5, \quad 6, \quad 7, \quad 3, \quad 2, \quad 1 \\ n &= 22, \quad 13, \quad 10, \quad -14, \quad -5, \quad -2. \end{aligned}$$

Of course, the values of m and n could be swapped; that is, we could have $(m,n) = (5,22)$ or $(m,n) = (22,5)$. This leads to a total of 12 unique solutions in integers.

Example 7. If $1/x + 1/y - 1/(xy) = 1$ where x and y are integers, how many ordered pairs (x,y) exist such that $x + y = 5$? [Walton 2013, Varsity]

(A) 2 **(B)** 3 **(C)** 10 **(D)** 24 **(E)** 25

Solution. Multiplying the equation through by xy gives us $y+x-1 = xy$. Then $xy - x - y = -1$. Using SFFT, we add 1 to both sides to get $xy - x - y + 1 = 0$, which factors into $(x-1)(y-1) = 0$. It follows that either $x = 1$ or $y = 1$. Since we also require $x + y = 5$, we have the solutions $(x,y) = (1,4)$ and $(x,y) = (4,1)$. There are two solutions. Thus, the answer is **A**. ◆

Example 8. If $a^2 + 2 = b^2 + 2014$, and a and b are positive integers, compute $\sqrt{ab}$. [Lassiter 2014, Varsity]

(A) $2\sqrt{503}$ **(B)** $6\sqrt{251}$ **(C)** $12\sqrt{251}$ **(D)** $6\sqrt{3114}$ **(E)** $12\sqrt{1757}$

Solution. We rewrite the equation as $a^2 - b^2 = 2012$. Then factoring yields $(a-b)(a+b) = 2012$. Since $4 \mid 2012$, we have $2012 = 2^2 \cdot 503$. Now we check whether 503 is divisible by a prime less than $\sqrt{503} < 23$. We can easily conclude that 503 is not divisible by 3, 5, 7, 11, or 13. Checking 17 and 19 (by long division) reveals that 503 must be prime. Hence, $(a-b)(a+b) = 2^2 \cdot 503$. Thus, we have the following possibilities for $a - b$ and $a + b$:

$$\begin{aligned} a-b &= 1 \quad 2 \quad 4 \\ a+b &= 2012 \quad 1006 \quad 503. \end{aligned}$$

These are the only possibilities because a and b are positive—this implies $a - b < a + b$. We effectively have three systems of equations. We solve these by adding the equations. We obtain, respectively, $2a = 2013$, $2a = 1008$, and

$2a = 507$. Only $2a = 1008$ gives a solution in integers; it is $a = 504$. It follows that $b = 504 - 2 = 502$. Therefore

$$\begin{aligned}\sqrt{ab} &= \sqrt{504 \cdot 502} = \sqrt{7 \cdot 8 \cdot 9 \cdot 2 \cdot 251} \\ &= \sqrt{9 \cdot 16 \cdot 7 \cdot 251} = \sqrt{144 \cdot 1757} = 12\sqrt{1757}.\end{aligned}$$

Thus, the answer is **E**. ◆

Exercises

8. Find $|a+b|$ if a and b are negative integers such that $ab + a + b = 96$. [Milton 2003, JV]

   (A) 30 **(B)** 45 **(C)** 96 **(D)** 98 **(E)** None of these

9. Find the number of ordered pairs of integers (a, b), $1 < a \leq 15$, such that $a^2 + b^2 - ab^2 - ab + b = 1$. [CSU 2016]

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Number Theoretic Functions

Using divisibility to find the prime factorization of a number is a good first step in many number theory problems. Once we have the prime factorization, we can find many things about the divisors. For instance, consider 24. Its divisors are simple to list: 1, 2, 3, 4, 6, 8, 12, and 24; there are 8 of them. The sum of the divisors is $1 + 2 + 3 + 4 + 6 + 8 + 12 + 24 = 60$. Now, let's relate this to the prime factorization: $24 = 2^3 \cdot 3$. The prime factorization of each of its divisors, in terms of the factors of 24, is

$$\begin{array}{llll} 1 = 2^0 \cdot 3^0, & 2 = 2^1 \cdot 3^0, & 4 = 2^2 \cdot 3^0, & 8 = 2^3 \cdot 3^0, \\ 3 = 2^0 \cdot 3^1, & 6 = 2^1 \cdot 3^1, & 12 = 2^2 \cdot 3^1, & 24 = 2^3 \cdot 3^1. \end{array}$$

In each divisor appear either zero, one, two, or three 2s, and either zero or one 3s. Hence, we have 4 options for including 2s and 2 options for including 3s, for a total of $4 \cdot 2 = 8$ possible divisors of 24. The numbers 4 and 2 can be gathered from the prime factorization; they are the powers on each prime increased by 1. This gives a reason for defining the following function.

τ is the Greek letter "tau".

Definition. Let n be a positive integer which has the prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i}$ where i is a positive integer, each p_i is a prime and each k_i is a positive integer. The number of divisors of n is the function $\tau(n)$ defined by $\tau(n) = (k_1 + 1)(k_2 + 1) \cdots (k_i + 1)$.

Recall the sum of the divisors of 24 was 60. Let's rearrange the sum:

$$\begin{aligned}1 + 2 + 3 + 4 + 6 + 8 + 12 + 24 &= 1 + 2 + 4 + 8 + 3 + 6 + 12 + 24 \\ &= 1 + 2 + 4 + 8 + 3(1 + 2 + 4 + 8) \\ &= (1 + 2 + 4 + 8)(1 + 3) = 15 \cdot 4 = 60.\end{aligned}$$

The prime factorization of 24 was $24 = 2^3 \cdot 3$, and in each set of parentheses is the sum of the powers of 2 and 3 up to those included in the prime factorization. This motivates defining the following function.

Definition. Let n be a positive integer which has the prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i}$ where i is a positive integer, each p_i is a prime, and each k_i is a positive integer. The sum of the divisors of n is the function $\sigma(n)$ defined by $\sigma(n) = (1 + p_1 + p_1^2 + \cdots + p_1^{k_1})(1 + p_2 + p_2^2 + \cdots + p_2^{k_2}) \cdots (1 + p_i + p_i^2 + \cdots + p_i^{k_i})$.

σ is the Greek letter "sigma".

In practice, the evaluation of $\sigma(n)$ can be made more efficient for primes with larger exponents. The factors of $\sigma(n)$ are all of the form $1 + p + p^2 + \cdots + p^k$, which is a geometric series with sum $(p^{k+1} - 1)/(p - 1)$.

The last function we define concerns numbers that are *relatively prime*. Two integers are relatively prime if they do not share any factors other than 1. Our last function returns the number of positive integers less than and relatively prime to an integer.

Some people use the term coprime *instead of relatively prime. But those people are relatively silly (or cosilly).*

Definition. Let n be a positive integer which has the prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i}$ where i is a positive integer, each p_i is a prime, and each k_i is a positive integer. The number of positive integers less than n and relatively prime to n is the function $\phi(n)$ defined by

ϕ is the Greek letter "phi".

$$\phi(n) = n \cdot \frac{p_1 - 1}{p_1} \cdot \frac{p_2 - 1}{p_2} \cdot \frac{p_3 - 1}{p_3} \cdots \frac{p_i - 1}{p_i}.$$

The function $\phi(n)$ is called *Euler's totient function*.

The word totient *is related to* quotient. *Both are derived from Latin:* quot, *for "how many" and* tot, *for "that many".*

Since $24 = 2^3 \cdot 3$, we have

$$\phi(24) = 24 \cdot \frac{2-1}{2} \cdot \frac{3-1}{3} = 24 \cdot \frac{1}{2} \cdot \frac{2}{3} = 8.$$

The 8 integers less than and relatively prime to 24 are 1, 5, 7, 11, 13, 17, 19, and 23. This formula for $\phi(n)$ works because we are taking a proportion of all numbers less than n. If we want to eliminate all integers less than 24 that share 2 as a factor, we will remove $1/2$ of them, since $1/2$ are even. From the 12 odd numbers left, $1/3$ are divisible by 3, but we want the proportion that isn't; that is, $2/3$ of those 12 odds are not divisible by 3. Hence, $\phi(24) = 24 \cdot 1/2 \cdot 2/3$.

Example 9. Let N be the number of factors of 2020 and let S be the sum of the factors of 2020. Compute S/N.

Solution. First, we find the prime factorization. Clearly $20 \mid 2020$; indeed, $2020 = 20 \cdot 101$, and 101 is prime. Thus, $2020 = 2^2 \cdot 5 \cdot 101$. Therefore $N = \tau(2020) = (2+1)(1+1)(1+1) = 3 \cdot 2 \cdot 2 = 12$, and $S = \sigma(2020) = (1 + 2 + 4)(1 + 5)(1 + 101) = 7 \cdot 6 \cdot 102$. Finally $S/N = 7 \cdot 6 \cdot 102/12 = 7 \cdot 51 = 357$. Thus, the answer is **357**. ◆

Exercises

10. Find the sum of the positive integral factors of 1998.

[GAC 2002, Comprehensive]

(A) 2562 **(B)** 3156 **(C)** 4560 **(D)** 6480 **(E)** None of these